Programlama -1

“Statistics”

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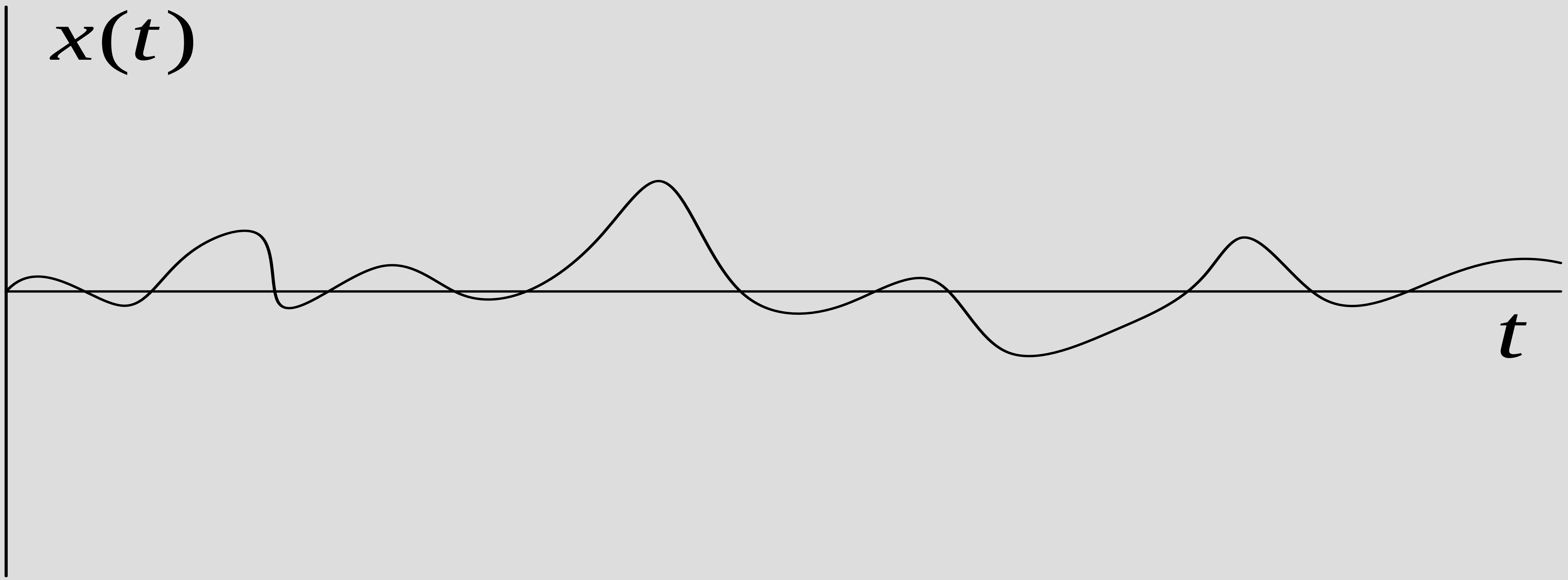
Statistics

Rastgele bir değişken, kesin davranışın tahmin edilemediği, ancak olası davranış terimleriyle tanımlanabilen bir değişkendir. Rastgele bir süreç, bir veya daha fazla rastgele değişken içeren herhangi bir süreçtir. Olasılık ve istatistik konuları, rastgele süreçler ve değişkenlerle ilgilenmek için matematiksel ve bilimsel bir metodolojiye dayanmaktadır.

Figure 1. A random variable.



Probability

Probability is associated with the very natural trend of a random event to follow a somewhat regular pattern *if the process is repeated a sufficient number of times.*

$$P(A_1) = \text{ probability that event 1 occurs}$$

$$P(A_2) = \text{ probability that event 2 occurs}$$

$$. \qquad\qquad . \qquad\qquad .$$

$$P(A_K) = \text{ probability that event } K \text{ occurs}$$

Probability

Assume that the first event occurs n_1 times, the second event n_2 times, and so on. The various probabilities are then defined as

$$P(A_1) = \lim_{n\to\infty} \frac{n_1}{n}$$

$$P(A_2) = \lim_{n\to\infty} \frac{n_2}{n}$$

$$\cdot \qquad\qquad\qquad \cdot$$

$$P(A_K) = \lim_{n\to\infty} \frac{n_K}{n}$$

Probabilities Properties

$$0 \leq P(A_k) \leq 1$$

$$\text{If } P(A_k) = 1, \text{ event is certain.}$$

$$\text{If } P(A_k) = 0, \text{ event will never occur.}$$

$$P(A_1) + P(A_2) + \ . \ . \ . \ + P(A_K) = 1$$

Example-1. For 52-card deck, determine probabilities of drawing (a) red card, (b) a heart, (c) an ace, (d) ace of spades, (e) ace of hearts or the ace of diamonds.

(a) $$P(R) = \frac{26}{52} = \frac{1}{2}$$

(b) $$P(H) = \frac{13}{52} = \frac{1}{4}$$

Example-1. Continuation.

(c) $$P(A) = \frac{4}{52} = \frac{1}{13}$$

(d) $$P(AH) = \frac{1}{52}$$

(e) $$P(E) = \frac{2}{52} = \frac{1}{26}$$

Probability Terminology

$P(A) =$ probability that A occurs.

$P(\bar{A}) =$ probability that A does not occur.

$P(A_1 + A_2) =$ probability that A_1 or A_2 occurs.

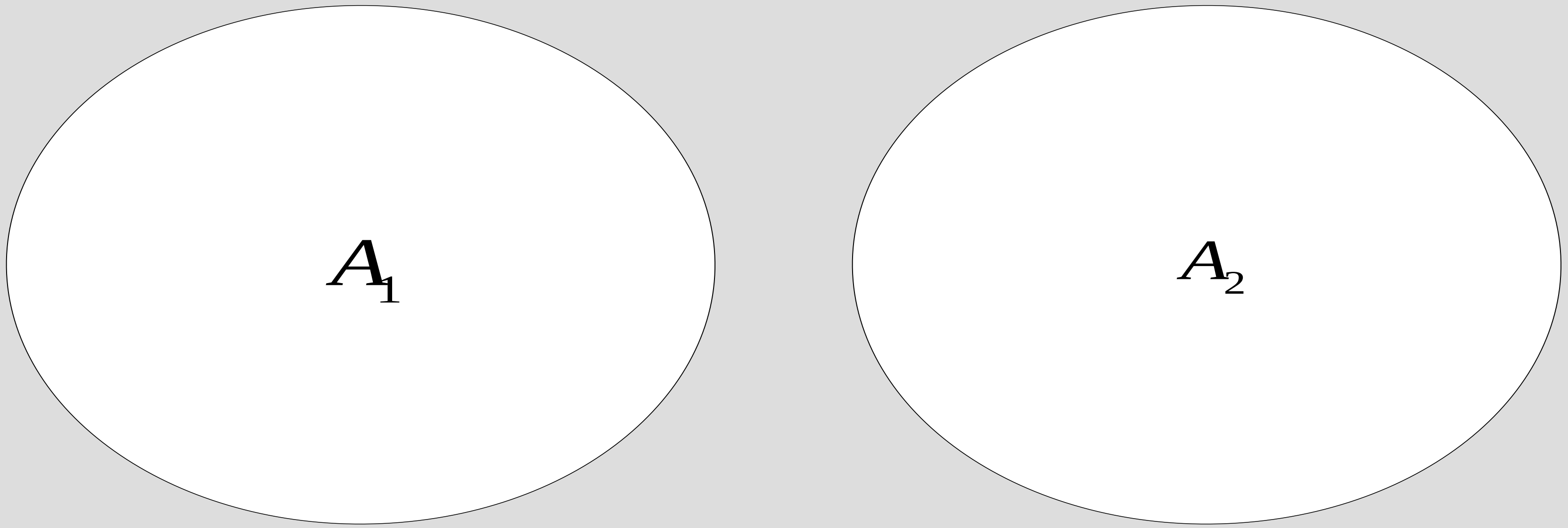
$P(A_1 A_2) =$ probability that A_1 and A_2 occur.

Mutual Exclusiveness

Let $P(A_1+A_2)$ represent the probability that either A_1 **or** A_2 occurs. Two events are mutually exclusive if

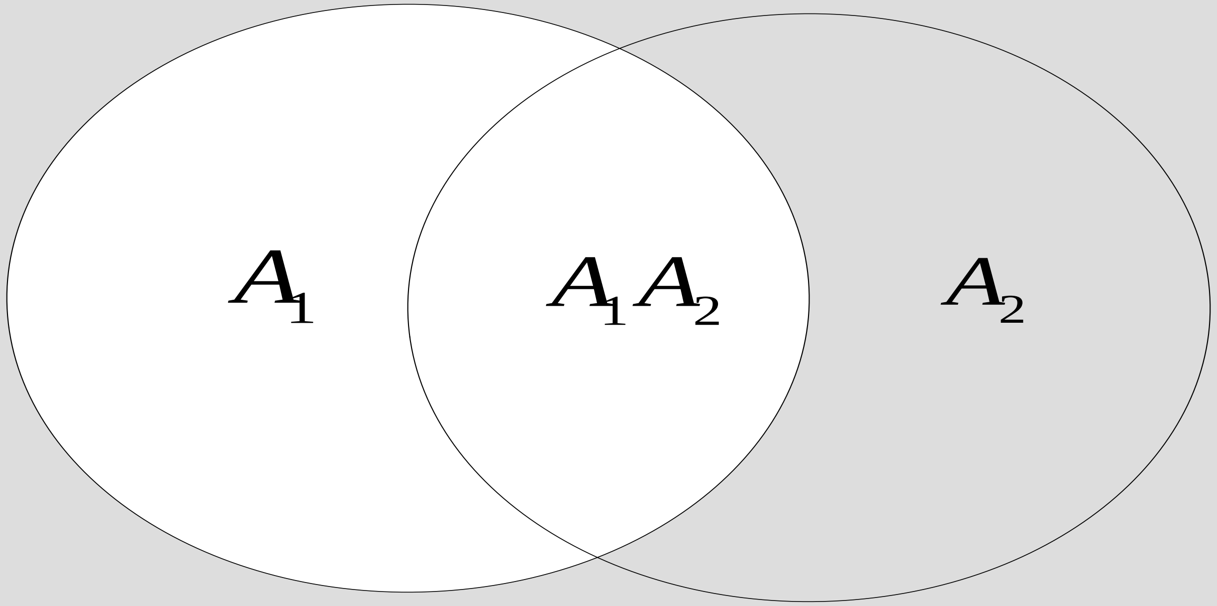
$$P(A_1 + A_2) = P(A_1) + P(A_2)$$

Figure 12-2(a). Events that are mutually exclusive.



(a)

Figure 12-2(b). Events that are not mutually exclusive.



(b)

Events that are not Mutually Exclusive

If the two events are not mutually exclusive, then the common area must be subtracted from the sum of the probabilities.

$$P(A_1 + A_2) = P(A_1) + P(A_2) - P(A_1 A_2)$$

Statistical Independence

Let $P(A_1 A_2)$ represent the probability that both A_1 **and** A_2 occur. Two events are statistically independent if

$$P(A_1 A_2) = P(A_1) P(A_2)$$

Conditional Probability

$P(A_2/A_1)$ is defined to mean "the probability that A_2 is true *given* that A_1 is true."

$$P(A_2 / A_1) = \frac{P(A_1 A_2)}{P(A_1)}$$

$$P(A_1 A_2) = P(A_1) P(A_2 / A_1)$$

Example 12-2. What is the probability that a single card will be an ace *or* a king?

$$P(A) = \frac{4}{52} = \frac{1}{13}$$

$$P(K) = \frac{4}{52} = \frac{1}{13}$$

$$P(A+K) = P(A) + P(K) = \frac{1}{13} + \frac{1}{13} = \frac{2}{13}$$

Example 12-3. What is the probability that a single card will be an ace *or* a red card?

$$P(A)=\frac{1}{13} \qquad P(R)=\frac{26}{52}=\frac{1}{2}$$

$$P(AR)=\frac{2}{52}=\frac{1}{26}$$

$$P(A+R)=P(A)+P(R)-P(AR)$$

$$=\frac{1}{13}+\frac{1}{2}-\frac{1}{26}=\frac{7}{13}$$

Example 12-4. Two cards are drawn from a deck. The first is replaced before the second is drawn. What is the probability that both will be aces?

$$P(A_1) = \frac{1}{13} \qquad P(A_2) = \frac{1}{13}$$

$$P(A_1 A_2) = P(A_1)P(A_2)$$

$$= \left(\frac{1}{13}\right)^2 = \frac{1}{169}$$

Example 12-5. Consider the same experiment but assume the first card is not replaced. What is the probability that both will be aces?

$$P(A_1) = \frac{1}{13} \qquad P(A_2 / A_1) = 3/51$$

$$P(A_1 A_2) = P(A_1) P(A_2 / A_1)$$

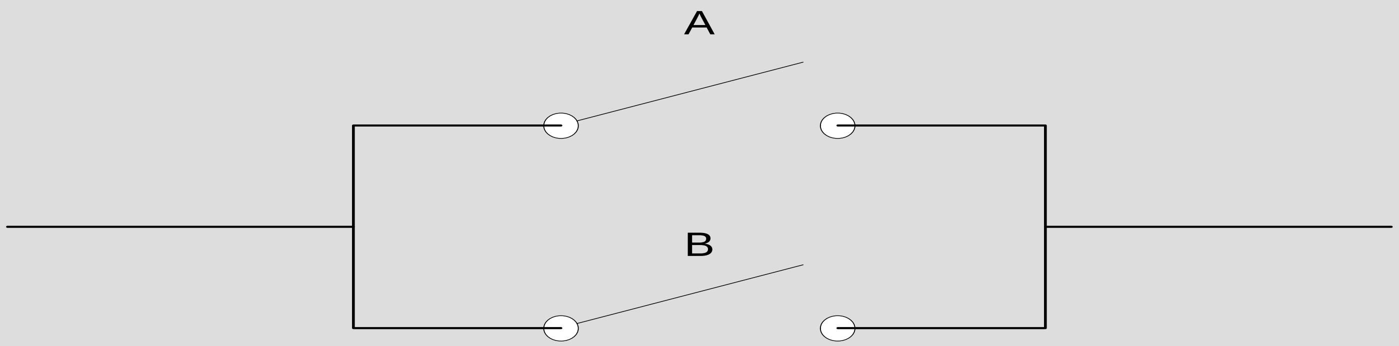
$$= \left(\frac{1}{13}\right) \times \left(\frac{3}{51}\right) = \frac{1}{221}$$

Example 12-6. The switches below are SI and only close 90% of the time when activated. Determine probability that both close.

$$P(A) = P(B) = 0.9$$

$$P(S) = P(AB) = P(A)P(B)$$
$$= (0.9)(0.9) = 0.81$$

Example 12-7. The switches are changed as shown. Determine the probability of success.



$$P(A) = P(B) = 0.9$$

$$P(S) = P(A+B) = P(A) + P(B) - P(AB)$$

$$P(S) = 0.9 + 0.9 - 0.81 = 0.99$$

Example 12-7. Alternate Solution.

$$P(\bar{A}) = P(\bar{B}) = 0.1$$

$$P(F) = P(\bar{A})P(\bar{B}) = (0.1)(0.1) = 0.01$$

$$P(S) = 1 - P(F) = 1 - 0.01 = 0.99$$

Discrete Statistical Functions

A *discrete variable* is one that can assume only a finite number of levels. For example, a binary signal can assume only two values: a logical 1 and a logical 0. To develop the concept, consider a random voltage $x(t)$ that can assume 4 levels. The values are listed below and a short segment is illustrated on the next slide.

$$x_1 = -5 \text{ V}, x_2 = 0 \text{ V}, x_3 = 5 \text{ V}, x_4 = 10 \text{ V}$$

Figure 12-5. Short segment of random discrete voltage.

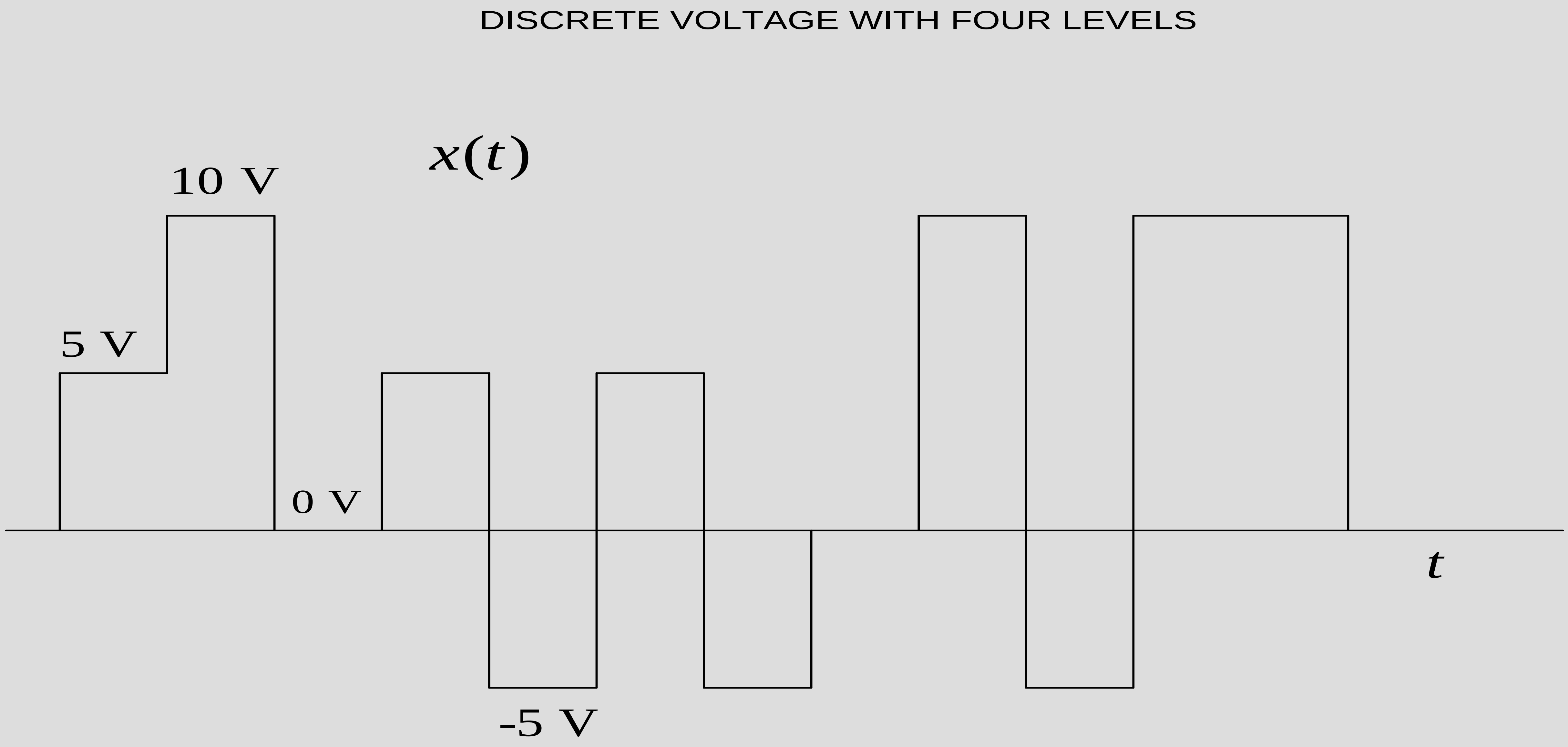


Figure 12-6. Number of samples of each voltage based on 100,000 total samples.

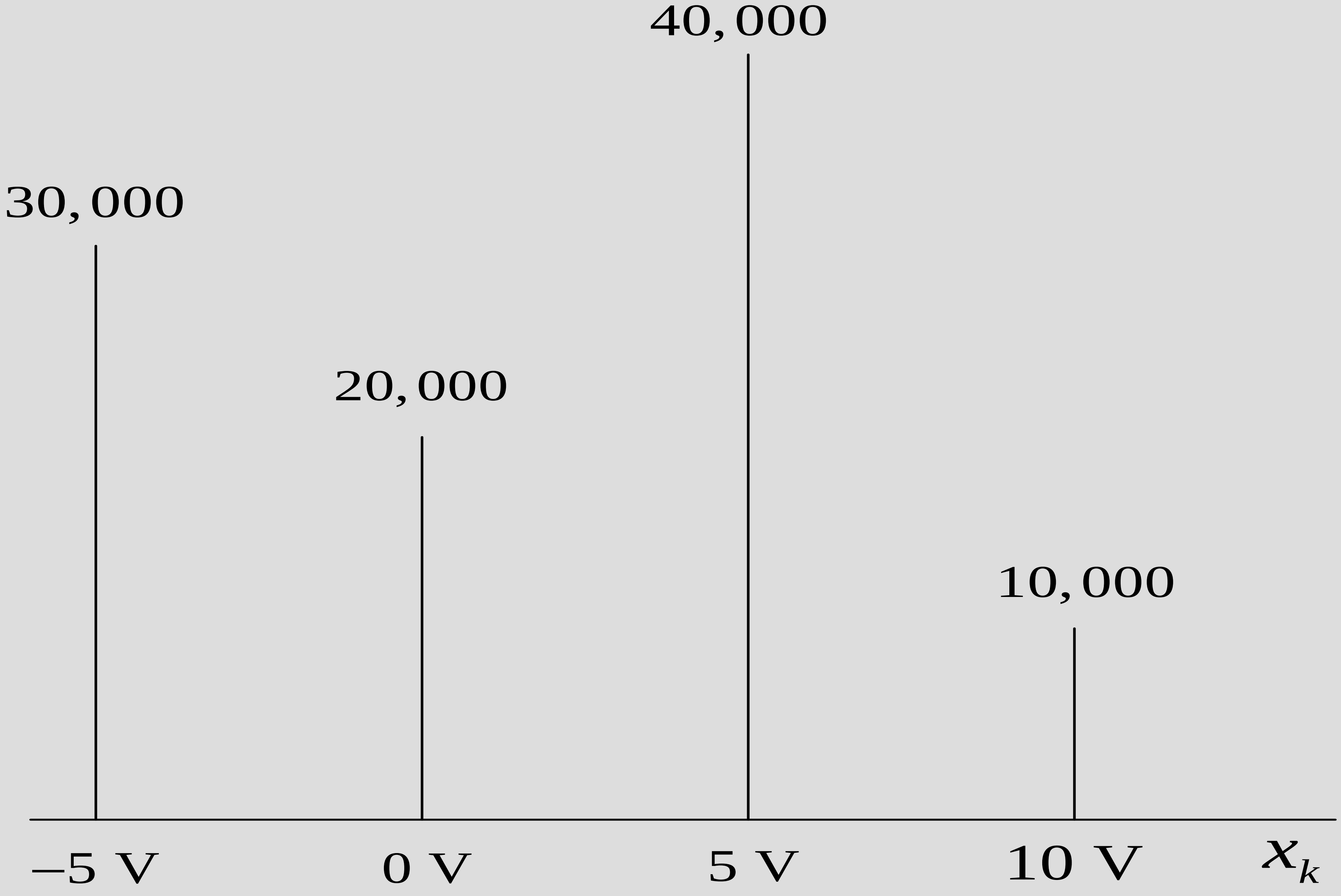
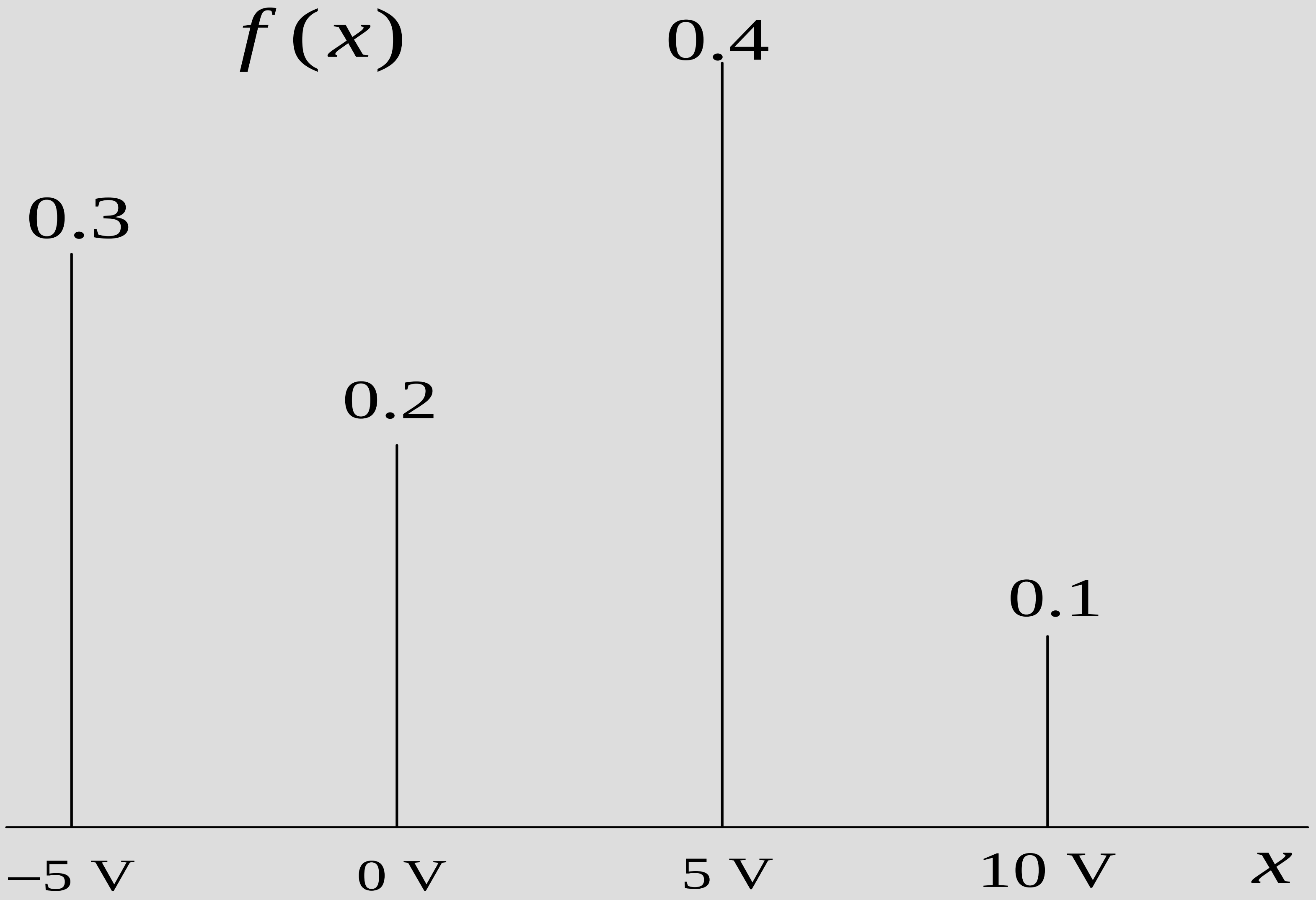


Figure 12-7. Probability density function (pdf) of random discrete voltage.



Probability Evaluations

The quantity *X* represents a random sample of a process. The expression $P(X=x)$ means **"the probability that a random sample of the process is equal to *x*."**

$$P(X=x)=f(x)$$

$$\sum_k f(x)=1$$

Probability Distribution Function $F(x)$

$$F(x) = P(X \leq x)$$

$$F(x_k) = \sum_{n=-\infty}^{k} f(x_n)$$

Example 12-8. For the pdf considered earlier, determine the probability values in the statements that follow.

$$P(X = 5) = f(5) = 0.4$$

$$P\left[(X = 0) + (X = 5)\right]$$

$$= P(X = 0) + P(X = 5)$$

$$= 0.2 + 0.4 = 0.6$$

Example 12-8. Continuation.

$$P(X > 0)$$

$$= P(X = 5) + P(X = 10)$$

$$= 0.4 + 0.1 = 0.5$$

$$P(X \geq 0)$$

$$= P(X = 0) + P(X = 5) + P(X = 10)$$

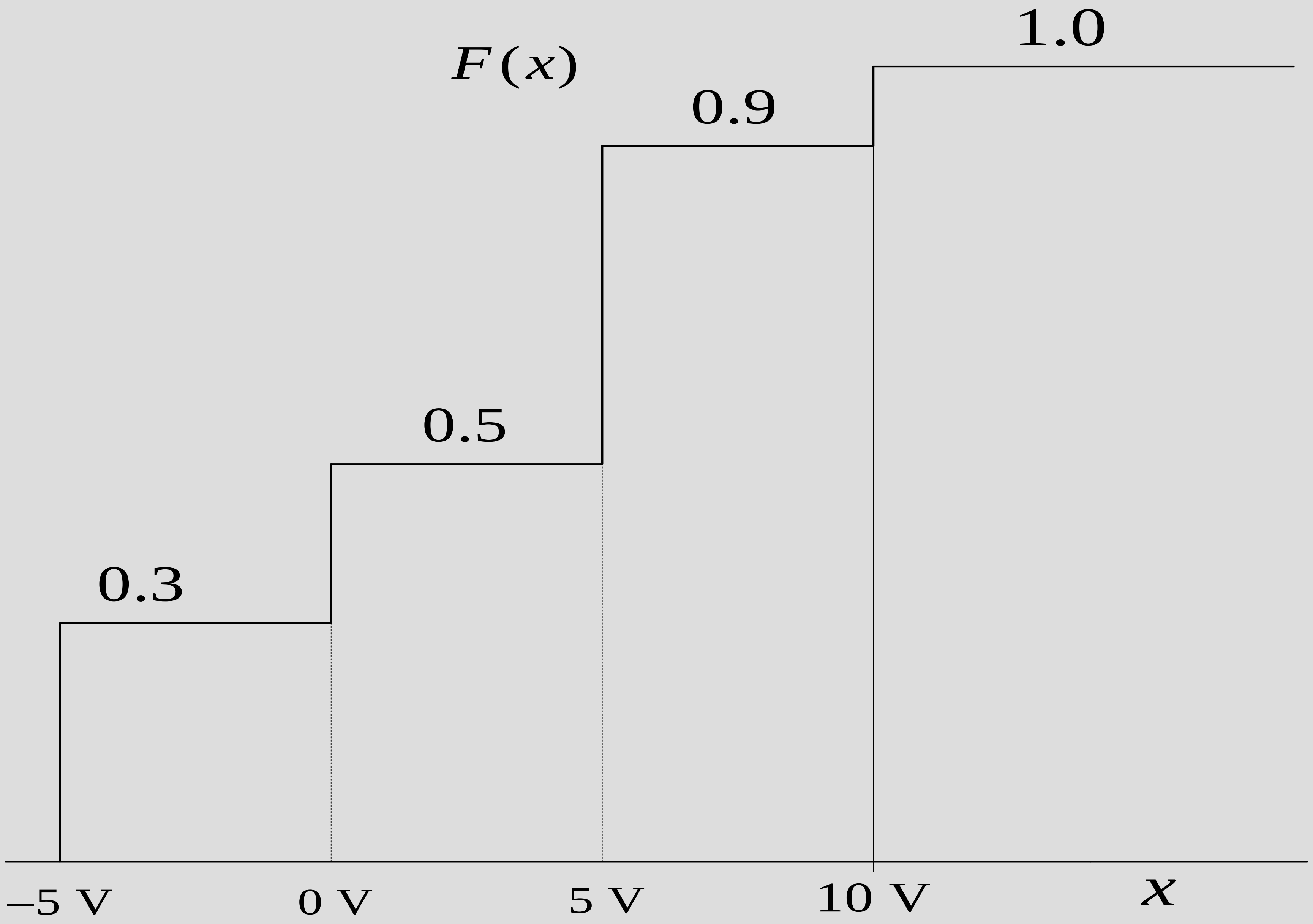
$$= 0.2 + 0.4 + 0.1 = 0.7$$

Example 12-8. Continuation.

$$P(-5 \leq X \leq 10)$$
$$= P(-5) + P(0) + P(5) + P(10)$$
$$= \sum_{k} f(x) = 1$$

$$P(X = 15) = 0$$

Example 12-9. Determine the probability distribution function of the random discrete voltage.



Statistical Averages of Discrete Variables.

In dealing with statistical processes, there is a difference between a *complete population* and a *sample* of a population insofar as parameter estimation is concerned. We will assume here that we are dealing with a complete population.

Expected Value or Expectation

$$E\left[g(x)\right] = \sum_{k} g(x) f(x)$$

Statistical Averages of Discrete Variables. Continuation.

Mean Value

$$\mu = E(x) = \sum_k x f(x)$$

Mean-Squared Value

$$E(x^2) = \sum_k x^2 f(x)$$

Root-Mean Square (RMS Value)

$$x_{rms} = \sqrt{E(x^2)}$$

Statistical Averages of Discrete Variables. Continuation.

Variance

$$\sigma^2 = E\left[(x-\mu)^2\right] = \sum_k (x-\mu)^2 f(x)$$

Alternate Formula for Variance

$$\sigma^2 = E(x^2) - \mu^2$$

Standard Deviation

$$\sigma = \sqrt{\sigma^2}$$

Example 12-10. For pdf of Example 12-8, determine (a) mean value, (b) mean-square value, (c) rms value, (d) variance, and (e) standard deviation.

$$\mu = \sum_k x f(x) = \sum_{k=1}^{4} x_k f(x_k)$$

$$= -5(0.3) + 0(0.2) + 5(0.4) + 10(0.1) = 1.5 \text{ V}$$

$$E(x^2) = \sum_k x^2 f(x) = \sum_{k=1}^{4} x_k^2 f(x_k)$$

$$= (-5)^2(0.3) + (0)^2(0.2) + (5)^2(0.4) + (10)^2(0.1)$$

$$= 27.5 \text{ V}^2$$

Example 12-10. Continuation.

$$x_{rms} = \sqrt{E(x^2)} = \sqrt{27.5} = 5.244 \text{ V}$$

$$\sigma^2 = E(x^2) - \mu^2 = 27.5 - (1.5)^2 = 25.25 \text{ V}^2$$

$$\sigma = \sqrt{\sigma^2} = \sqrt{25.25} = 5.025 \text{ V}$$

Binomial Probability Density Function

$$p = \text{ probability that A occurs in a given trial}$$

$$q = \text{ probability that B occurs in a given trial}$$

$$q = 1 - p$$

Consider the probability that in four trials, A will occur exactly twice. The different combinations are as follows:

AABB ABAB ABBA BBAA BABA BAAB

Combinations

The number of combinations of *n* trials with exactly *x* occurrences of one of the two outcomes, where *x* is a non-negative integer no greater than *n*, is given by

$$C_x^n = \frac{n!}{x!(n-x)!}$$

Binomial PDF

$$f(x) = C_x^n p^x q^{n-x} = \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$P(X = x) = f(x) = \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$0 \le x \le n$$

Example 12-11. An unbiased coin is flipped 3 times. What is the probability of getting exactly one head?

$$f(1) = C_1^3 (0.5)^1 (0.5)^2$$

$$= \frac{3!}{(1!)(2!)} (0.5)^1 (0.5)^2$$

$$= 3(0.5)^3 = 0.375$$

Example 12-12. For previous example, what is the probability of getting *at least* one head?

$$P(S) = f(1) + f(2) + f(3)$$

$$= \frac{3!}{(1!)(2!)}(0.5)^1(0.5)^2 + \frac{3!}{(2!)(1!)}(0.5)^2(0.5)^1$$

$$+ \frac{3!}{(3!)(0!)}(0.5)^1(0.5)^2$$

$$= 3(0.5)^3 + 3(0.5)^3 + (0.5)^3$$

$$= 0.375 + 0.375 + 0.125 = 0.875$$

Example 12-12. Alternate Approach.

$$P(\bar{H}) = P(X = 0) = f(0)$$

$$= \frac{3!}{(0!)(3!)}(0.5)^0(0.5)^3 = 0.125$$

$$P(H) = 1 - P(\bar{H})$$

$$= 1 - 0.125 = 0.875$$

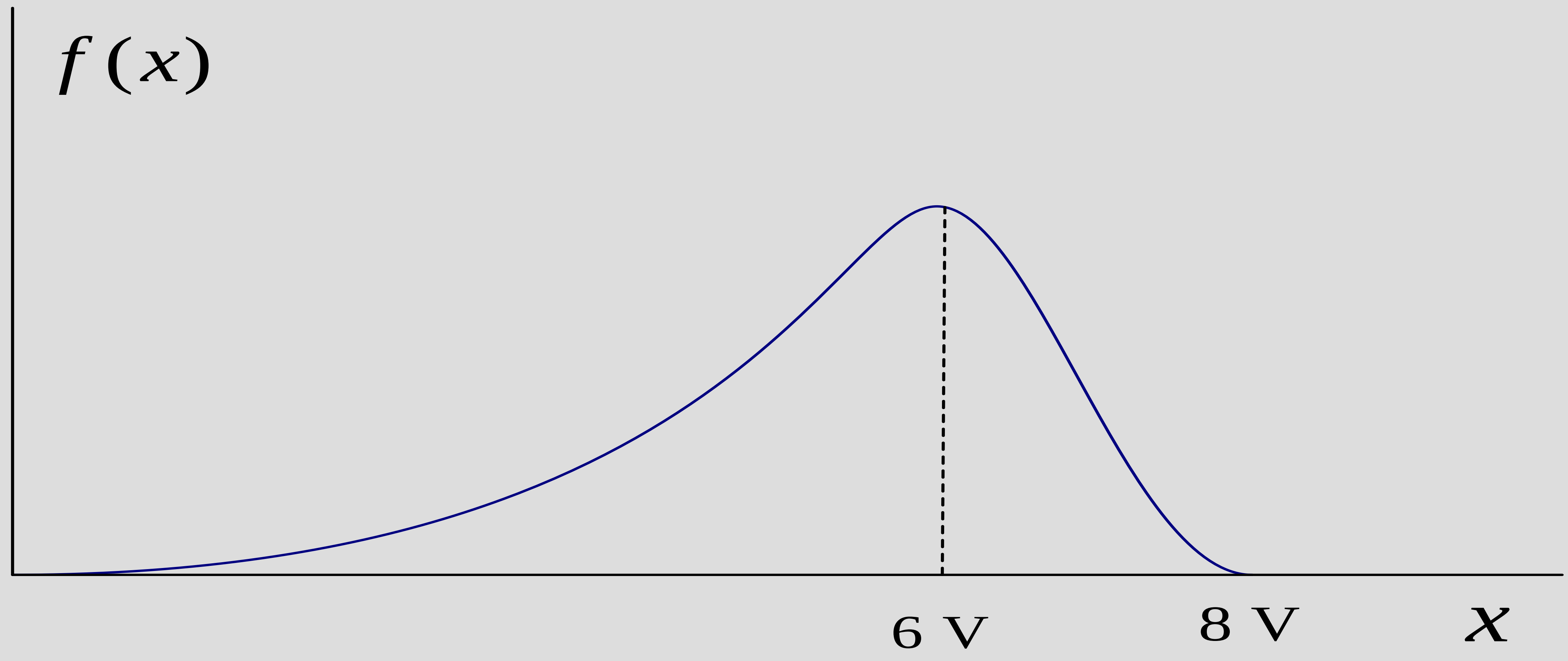
Continuous Statistical Functions

If the random variable *x* is continuous over a domain, it can be described by a continuous pdf. The probability of a sample assuming an **exact** value is 0.

$$P(x_1 \le X \le x_2) = \int_{x_1}^{x_2} f(x)dx$$

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

Figure 12-9. Typical pdf of a continuous variable.



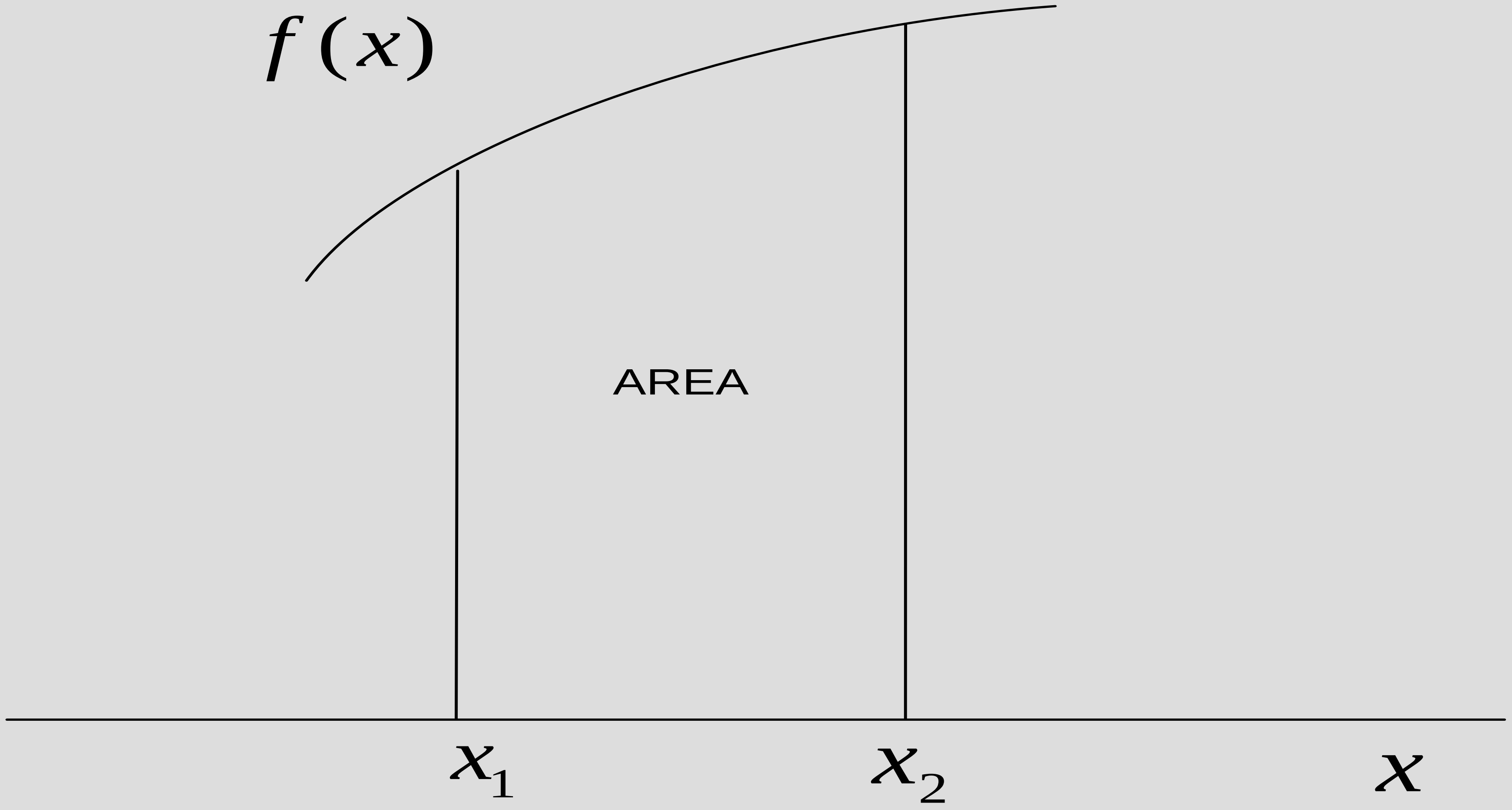
Probability Distribution Function $F(x)$

$$P(X \le x) = F(x)$$

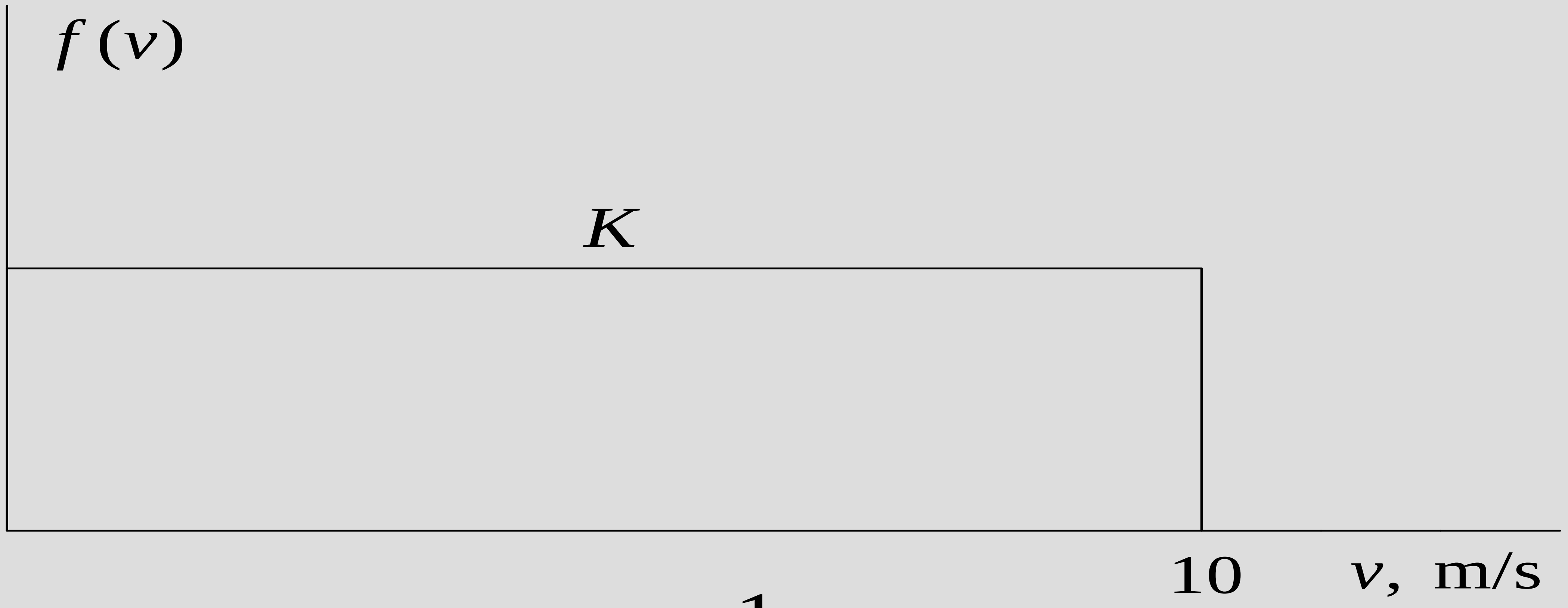
$$F(x) = \int_{-\infty}^{x} f(u)\,du$$

$$P(x_1 \le X \le x_2) = F(x_2) - F(x_1)$$

Figure 12-10. Probability of a random sample lying within a domain is the area under the pdf curve between the limits.



Example 12-13. Determine *K* below.



$$10K = 1 \qquad K = \frac{1}{10} = 0.1$$

$$f(v) = 0.1 \text{ for } 0 \le v \le 10 \text{ m/s}$$
$$= 0 \text{ elsewhere}$$

Example 12-14. For the pdf of the last example, determine the probability values in the statements that follow. Refer to Figure 12-12 if necessary.

$$P(V = 5 \text{ m/s}) = 0$$

$$P(4.9 \le V \le 5.1)$$

$$= 0.1 \times (5.1\text{-}4.9) = 0.02$$

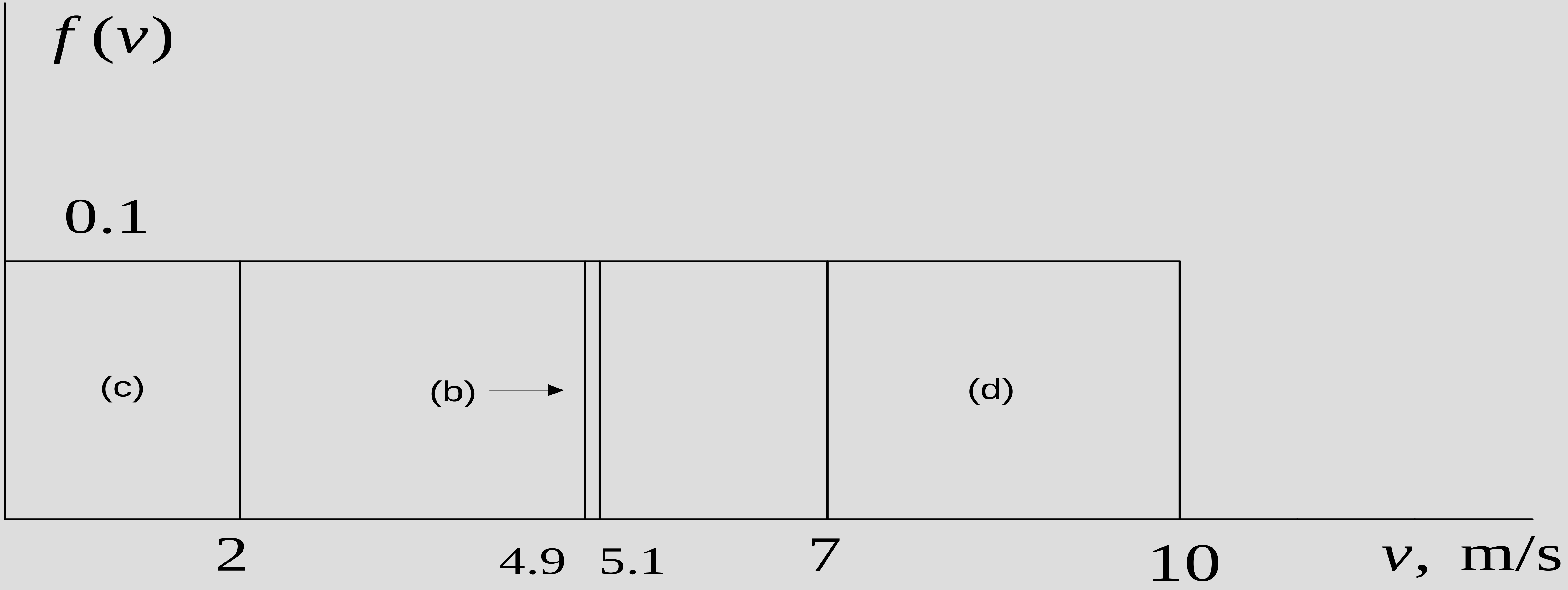
Example 12-14. Continuation.

$$P(V \leq 2) = 0.1 \times 2 = 0.2$$

$$P(V \geq 7) = 0.1 \times (10\text{-}7) = 0.3$$

$$P(0 \leq V \leq 10) = 1$$

Figure 12-12. Various areas in Example 12-14.



Example 12-15. Determine the probability distribution function for the uniform pdf of Example 12-14.

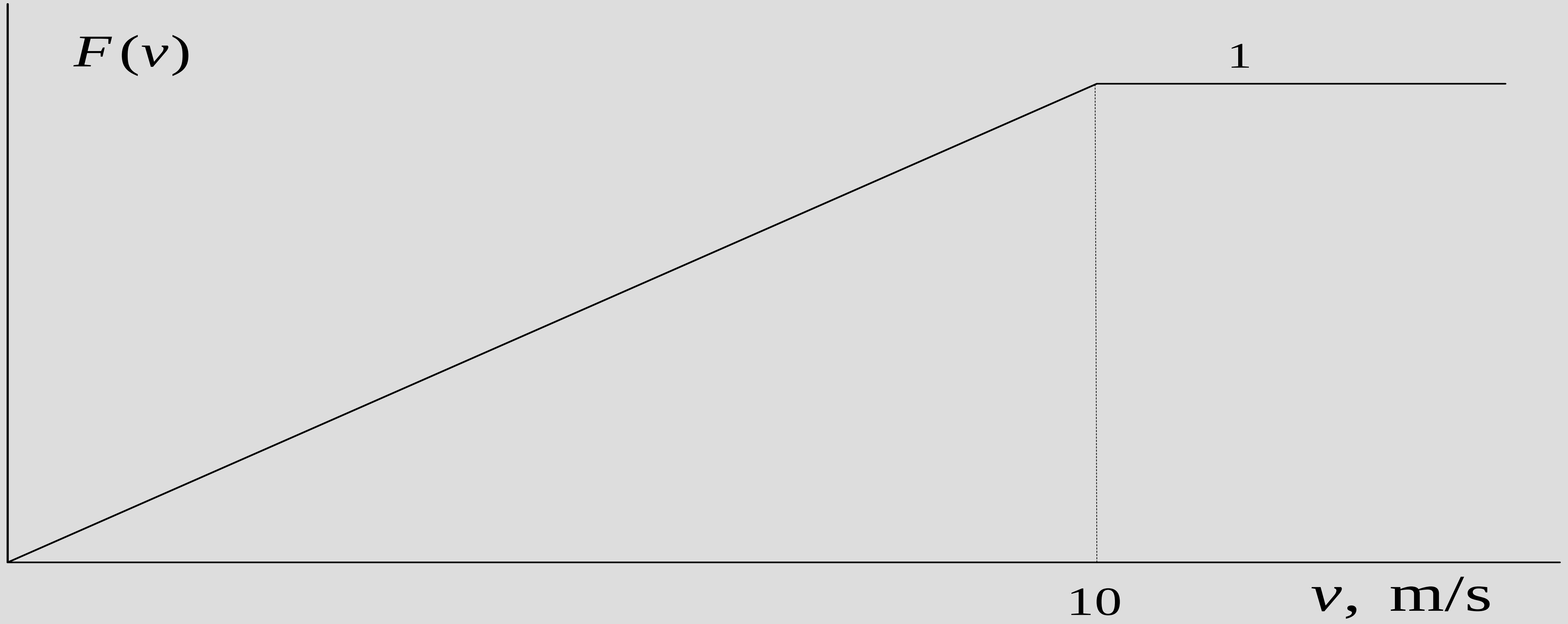
$$F(v) = \int_0^v f(u)du$$

$$= \int_0^v (0.1)du$$

$$= 0.1u\Big]_0^v$$

$$= 0.1v \quad \text{for} \quad 0 \le v \le 10$$

Figure 12-13. Probability distribution function for Example 12-15.



Statistical Averages of Continuous Variables.

With discrete variables, the averages are determined by summations. With continuous variables, the averages are determined by integrals.

Expected Value or Expectation

$$E\left[g(x)\right] = \int_{-\infty}^{\infty} g(x) f(x) dx$$

Statistical Averages of Continuous Variables. Continuation.

Mean Value

$$\mu = E(x) = \int_{-\infty}^{\infty} x f(x) dx$$

Mean-Squared Value

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$

Root-Mean Square (RMS Value)

$$x_{rms} = \sqrt{E(x^2)}$$

Statistical Averages of Discrete Variables. Continuation.

Variance

$$\sigma^2 = E\left[(x-\mu)^2\right] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$$

Alternate Formula for Variance

$$\sigma^2 = E(x^2) - \mu^2$$

Standard Deviation

$$\sigma = \sqrt{\sigma^2}$$

Example 12-16. For uniform pdf of previous examples, determine (a) mean value, (b) mean-square value, (c) total rms value, (d) variance, and (e) standard deviation.

$$\mu = \int_{-\infty}^{\infty} v f(v) dv = \int_{0}^{10} 0.1 v dv = \left. \frac{0.1 v^2}{2} \right]_0^{10} = 5 \text{ m/s}$$

$$E(v^2) = \int_{-\infty}^{\infty} v^2 f(v) dv = \int_{0}^{10} 0.1 v^2 dv = \left. \frac{0.1 v^3}{3} \right]_0^{10} = 33.33 \text{ m}^2/\text{s}^2$$

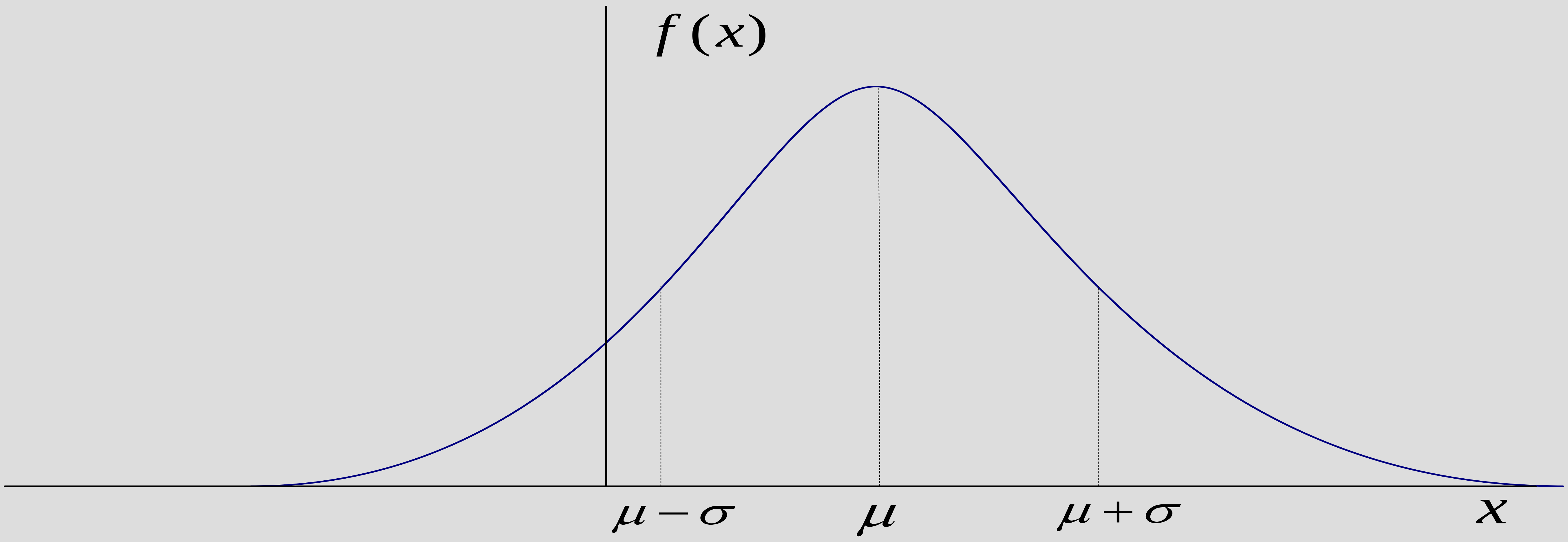
Example 12-16. Continuation.

$$v_{rms} = \sqrt{33.33} = 5.774 \text{ m/s}$$

$$\sigma^2 = E(v^2) - \left(\mu\right)^2 = 33.33 - (5)^2 = 8.333 \text{ m}^2/\text{s}^2$$

$$\sigma = \sqrt{8.333} = 2.887 \text{ m/s}$$

Figure 12-14. Gaussian Probability Density Function.



Mathematical Properties of Gaussian PDF

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} \qquad \text{for } -\infty < x < \infty$$

$$P(x_1 \le X \le x_2) = \int_{x_1}^{x_2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} \, dx$$

Normalized Gaussian PDF

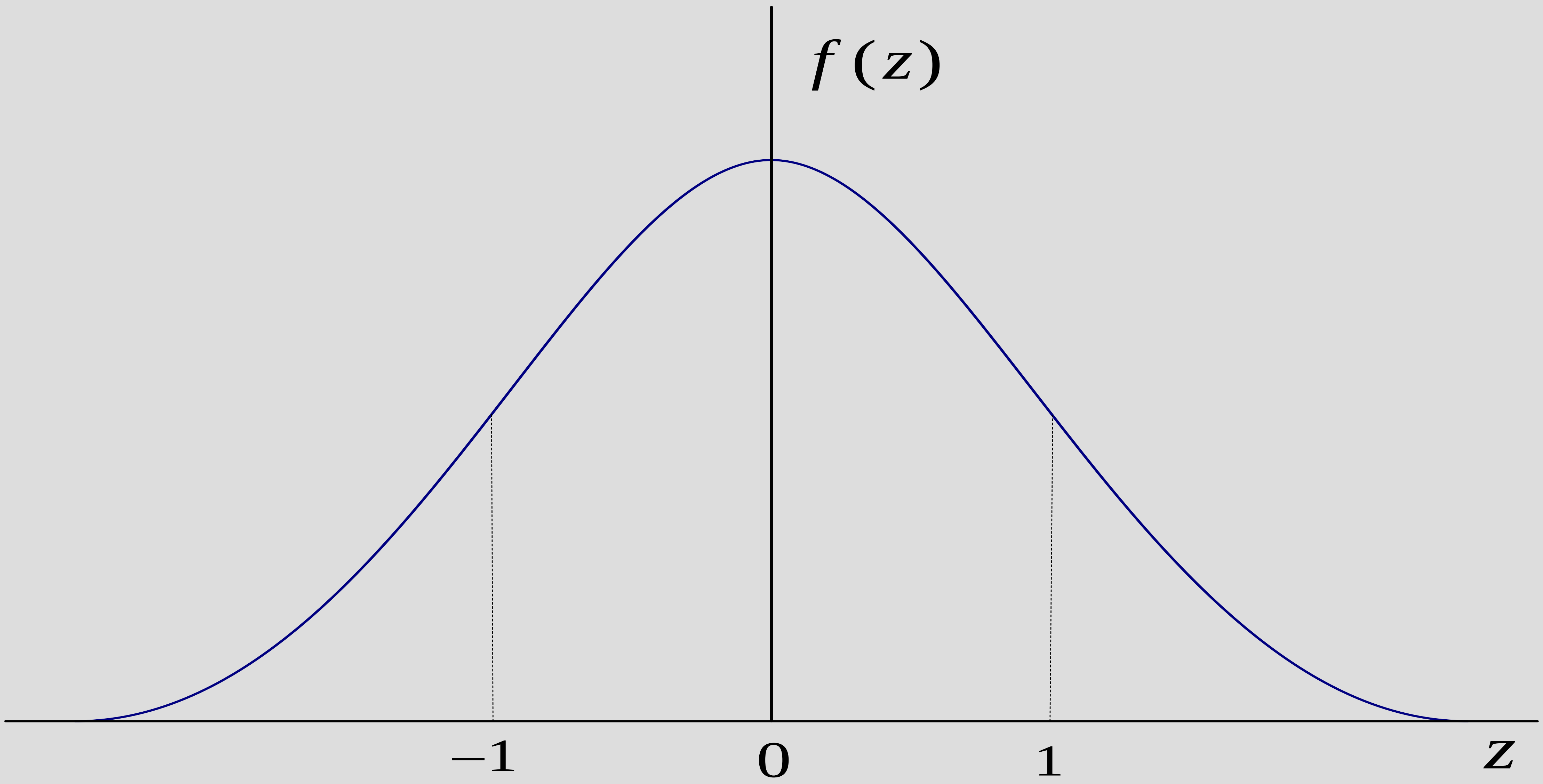
$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \qquad \text{for } -\infty < z < \infty$$

$$\mu = 0 \text{ and } \sigma^2 = 1$$

$$z = \frac{x - \mu}{\sigma}$$

$$x = \sigma z + \mu$$

Figure 12-15. Normalized Gaussian PDF.



Gaussian Probability Distribution Function

$$F(z) = \int_{-\infty}^{z} f(u)du$$

$$P(z_1 \leq Z \leq z_2) = F(z_2) - F(z_1)$$

$$F(-\infty) = 0$$

$$F(0) = 0.5$$

$$F(\infty) = 1$$

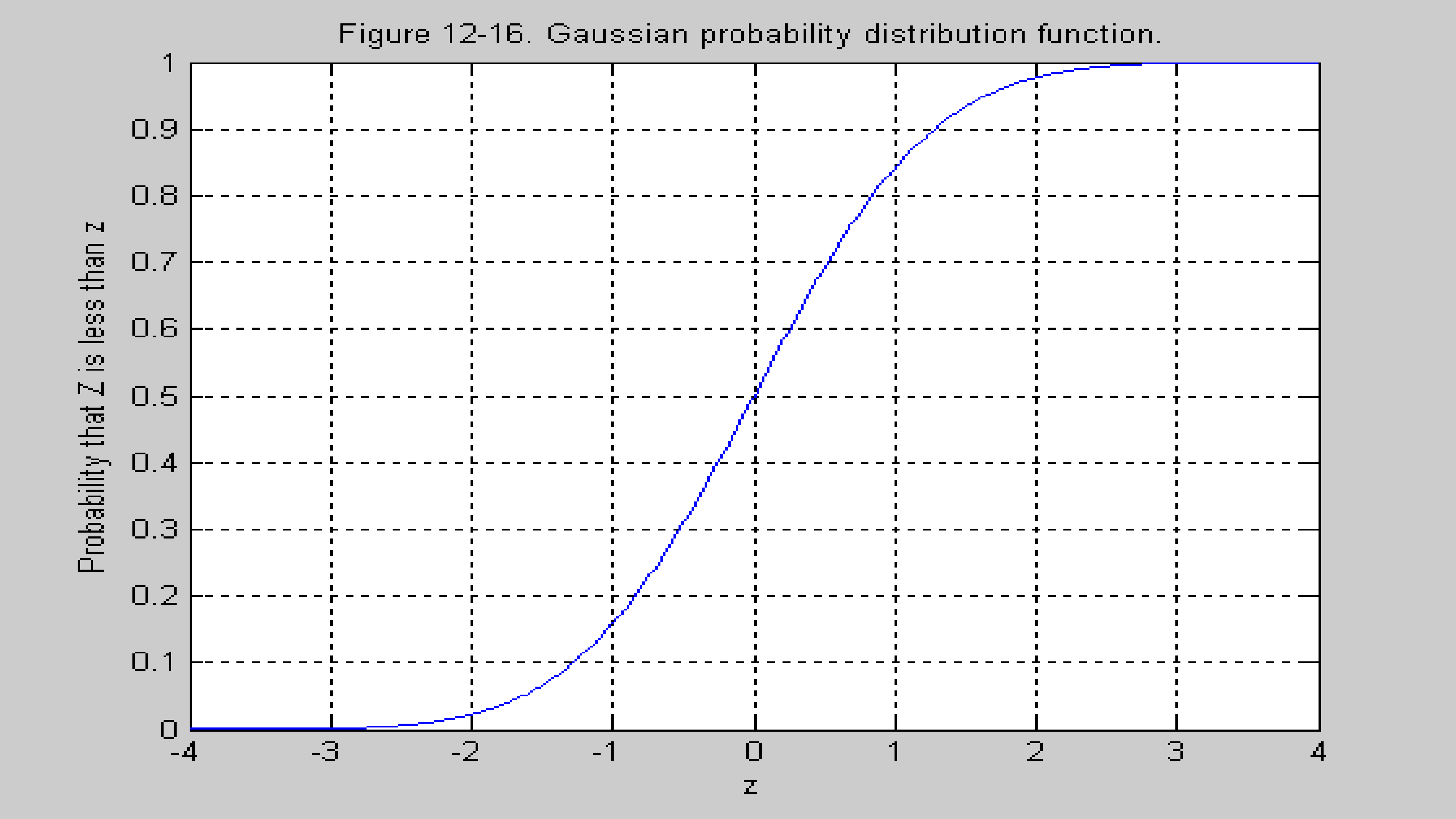


Figure 12-16. Gaussian probability distribution function.
Probability that Z is less than z
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-4
-3
-2
-1
0
1
2
3
4
z

Example 12-17. In this example, the normalized gaussian pdf will be used to evaluate various probabilities that will be listed on the next several slides. Table 12-2 in the text will be used to determine the values required in the computations.

Example 12-17. Continuation.

$$P(Z \geq 0) = P(0 \leq Z \leq \infty)$$

$$= F(\infty) - F(0) = 1 - 0.5 = 0.5$$

$$P(1 \leq Z \leq 3) = F(3) - F(1)$$

$$= 0.998650 - 0.841345 = 0.1573$$

$$P(-1 \leq Z \leq 2) = F(2) - F(-1)$$

$$= 0.977250 - 0.158655 = 0.8186$$

Example 12-17. Continuation.

$$P(-2 \le Z \le 2) = F(2) - F(-2)$$

$$= 0.977250 - 0.022750 = 0.9545$$

Alternately,

$$P(-2 \le Z \le 2) = 2P(0 \le Z \le 2)$$

$$= 2\left[F(2) - F(0)\right]$$

$$= 2\left[0.977250 - 0.5\right] = 0.9545$$

$$P(|Z| \le 2) = P(-2 \le Z \le 2) = 0.9545$$

Example 12-17. Continuation.

$$P(Z \geq 2) = P(2 \leq Z \leq \infty) = F(\infty) - F(2)$$

$$= 1 - 0.97725 = 0.02275$$

$$P(|Z| \geq 2) = P\left[(Z \geq 2) + (Z \leq -2)\right]$$

$$= 2P(Z \geq 2) = 2(0.02275) = 0.0455$$

Alternately,

$$P(|Z| \geq 2) = 1 - P(|Z| \leq 2) = 1 - 0.9545 = 0.0455$$

Example 12-18. A random voltage $x(t)$ is gaussian distributed with a dc value of 0 and an rms value of 5 V. Determine various probabilities listed in the steps that follow.

$$z=\frac{x-\mu}{\sigma}=\frac{x-0}{\sigma}=\frac{x}{5}$$

$$P(X \geq 10 \text{ V})=P(Z \geq 2)=F(\infty)-F(2)$$

$$=1-0.977250=0.02275$$

Example 12-18. Continuation.

$$P(|X| \geq 10) = 2P(Z \geq 2)$$
$$= 2 \times 0.02275 = 0.0455$$

$$P(|X| \leq 15 \text{ V}) = P(|Z| \leq 3)$$
$$= 2P(0 \leq Z \leq 3)$$
$$= 2 \times [0.998650 - 0.5] = 0.9973$$

Example 12-19. A random force has a gaussian distribution with a mean value of 10 N and standard deviation of 5 N. Determine the probability that a sample exceeds 20 N.

$$z = \frac{x - \mu}{\sigma} = \frac{x - 10}{5}$$

$$z = \frac{20 - 10}{5} = 2$$

Example 12-19. Continuation.

$$
\begin{aligned}
P(X \geq 20) & \\
&= P(Z \geq 2) \\
&= F(\infty) - F(2) \\
&= 1 - 0.97725 \\
&= 0.02275
\end{aligned}
$$

Sampling Statistics

In all cases thus far, it has been assumed that the statistics of the *complete population* were known. In many applications, only a *sample* of the population can be obtained.

Parameter	True	Sample
Mean	μ	$\overline{x}$
Mean-Square	$E(x^2)$	$\overline{x^2}$
Variance	σ^2	s^2
Standard Deviation	σ	s

Sample Parameters

Sample Mean

$$\overline{x} = \frac{1}{n}\sum_{k=1}^{n} x_k$$

Sample Mean-Square Value

$$\overline{x^2} = \frac{1}{n}\sum_{k=1}^{n} x_k^2$$

Sample Parameters (Continued)

Sample Variance

$$s^2 = \frac{1}{n-1}\sum_{k=1}^{n}(x_k - \bar{x})^2$$

or $$s^2 = \frac{1}{n-1}\left[\sum_{k=1}^{n} x_k^2 - \frac{\left(\sum_{k=1}^{n} x_k\right)^2}{n}\right]$$

Sample Standard Deviation

$$s = \sqrt{s^2}$$

Other Sample Statistical Parameters

Median. The median is the midpoint value.

Mode. The mode is the most frequently occurring value.

Range. The range is the largest value minus the smallest value.

Example 12-20. In a large class, a sample of 25 grades taken at random resulted in the values below. Determine the sample statistics.

50 52 56 60 63 68 70 73 76 76 76 77 79 81 82 82 83 85 88 89 89 90 92 94 96

$$\overline{x} = \frac{1}{n}\sum_{k=1}^{n} x_k = \frac{1}{25}\sum_{k=1}^{25} x_k = \frac{1927}{25} = 77.08$$

$$\overline{x^2} = \frac{1}{n}\sum_{k=1}^{n} x_k^2 = \frac{1}{25}\sum_{k=1}^{25} x_k^2 = \frac{152,585}{25} = 6103.40$$

Example 12-20. Continuation.

$$s^2 = \frac{1}{n-1}\sum_{k=1}^{n}(x_k - \bar{x})^2 = \frac{1}{24}\sum_{n=1}^{24}(x_k - 77.08)^2$$

$$= \frac{4051.84}{24} = 168.83$$

alternate method

$$s^2 = \frac{1}{n-1}\left[\sum_{k=1}^{n} x_k^2 - \frac{\left(\sum_{k=1}^{n} x_k\right)^2}{n}\right] = \frac{1}{24}\left[\sum_{k=1}^{25} x_k^2 - \frac{\left(\sum_{k=1}^{25} x_k\right)^2}{25}\right]$$

$$= \frac{1}{24}\left[152,585 - \frac{(1927)^2}{25}\right] = \frac{1}{24}[4051.84] = 168.83$$

Example 12-20. Continuation.

$$s = \sqrt{s^2} = \sqrt{168.83} = 12.99$$

$$\text{median} = 79$$

$$\text{mode} = 76$$

$$\text{range} = 96 - 50 = 46$$

MATLAB Statistical Parameters

Function	Format
Sample Mean	mean(x)
Sample Median	median(x)
Sample Standard Deviation	std(x)
Maximum Value	max(x)
Minimum Value	min(x)
Sum of all values	sum(x)
Product of all values	product(x)
Sort in ascending order	sort(x)

Example 12-21. Use MATLAB to solve Example 12-20.

```
>> xa = [50 52 56 60 63];
>> xb = [68 70 73 76 76];
>> xc = [76 77 79 81 82];
>> xd = [82 83 85 88 89];
>> xe = [89 90 92 94 96];


>> x = [xa xb xc xd xe];
```

Example 12-21. Continuation.

```
>> xmean = mean(x)
xmean =
   77.0800

>> xmedian = median(x)
xmedian =
    79
```

Example 12-21. Continuation.

```
>> xstd = std(x)
xstd =
   12.9933

>> xmean_square = mean(x.^2)
xmean_square =
  6.1034e+003
```

Gaussian Probabilities Using the Error Function

$$\operatorname{erf}(x) = \sqrt{\frac{2}{\pi}} \int_0^x e^{-u^2} du$$

$$F(z) = P(Z \le z) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right)\right]$$

$$P(z_1 \le Z \le z_2) = \frac{1}{2}\left[\operatorname{erf}\left(\frac{z_2}{\sqrt{2}}\right) - \operatorname{erf}\left(\frac{z_1}{\sqrt{2}}\right)\right]$$

Complementary Error Function

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$$

$$P(Z \geq z) = \frac{1}{2}\operatorname{erfc}\left(\frac{z}{\sqrt{2}}\right)$$

Summary of MATLAB Error Function Commands

Error Function: **erf(x)**

Inverse Error Function: **erfinv(x)**

Complementary Error Function: **erfc(x)**

Inverse Complementary Error Function:
erfcinv(x)

Inverse Operations with MATLAB

To find the value of z such that $P(Z \leq z) = p$, the following MATLAB command is used:

```
>> z = sqrt(2)*erfinv(2*p - 1)
```

To find the value of z such that $P(Z \geq z) = p$, the following MATLAB command is used:

```
>> z = sqrt(2)*erfcinv(2*p)
```

Example 12-22. Rework the first four parts of Example 12-17 using MATLAB.

```
>> k = sqrt(2);
```

(a) $P(Z \geq 0)$

```
>> pa=.5*(erf(10/k) - erf(0))
pa =
    0.5000
```

(b) $P(1 \leq Z \leq 3)$

```
>> pb = 0.5*(erf(3/k) - erf(1/k))
pb =
```

Example 12-22. Continuation.

(c) $P(-1 \leq Z \leq 2)$

```
>> pc = 0.5*(erf(2/k) - erf(-1/k))
pc =
    0.8186
```

(d) $P(-2 \leq Z \leq 2)$

```
>> pd = 0.5*(erf(2/k) - erf(-2/k))
pd =
    0.9545
```

Example 12-23. An internet line has noise with a dc value of 0 and an rms value of 2 V. An error occurs if the noise exceeds a threshold. Determine threshold for $p = 10^{-5}$.

```
>> z = sqrt(2)*erfcinv(2*1e-5)
z =
    4.2649
```

$$z = \frac{x-\mu}{\sigma} = \frac{x-0}{2} = \frac{x}{2}$$

$$x = 2z = 2 \times 4.2649 = 8.53\ \text{V}$$

Parameter	Continuous PDF	Discrete PDF	Sample
Mean	$\mu = \int_{-\infty}^{\infty} x f(x) dx$	$\mu = \sum_k x_k f(x_k)$	$\overline{x} = \frac{1}{n}\sum_{k=1}^{n} x_k$
Mean-Square	$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$	$E(x^2) = \sum_k x_k^2 f(x_k)$	$\overline{x^2} = \frac{1}{n}\sum_{k=1}^{n} x_k^2$
Root-Mean-Square	$x_{rms} = \sqrt{E(x^2)}$	$x_{rms} = \sqrt{E(x^2)}$	$x_{rms} = \sqrt{\overline{x^2}}$
Variance	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx$ $= E(x^2) - \mu^2$	$\sigma^2 = \sum_k (x_k - \mu)^2 f(x_k)$ $= E(x^2) - \mu^2$	$s^2 = \frac{1}{n-1}\sum_{k=1}^{n} (x_k - \overline{x})^2$
Standard Deviation	$\sigma = \sqrt{\sigma^2}$	$\sigma = \sqrt{\sigma^2}$	$s = \sqrt{s^2}$